Monte Carlo Minimization For Sequential Control

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Abstract

Sequential updating solutions to optimal control problems are typically not available in closed form. Here we present a method of Monte Carlo calculation of sequential updating solutions by simulating realizations from the predictive distribution of model parameters by Markov chain Monte Carlo and approximating the predictive expected loss (p. e. l.) by averaging over the simulations. The minimizer of the approximate p. e. l. is taken to approximate the exact p. e. l. The approximate minimizer is shown to converge to the exact minimizer under very weak regularity conditions (mere continuity of the loss function), and is shown to be asymptotically normal under stronger conditions. Examples are given from the problem of controlling a linear regression model with autoregressive responses (ARX) and from dynamic input-output models using a variety of loss functions.

1 Introduction

Control is a common problem in many fields. For example, in a medical application, one may want to control a physiological variable for a patient with some chronic disease by regulating the dosage level of a given drug. The regulated dosage level may depend on a patient's current physiological level and on other controllable or uncontrollable variables. A statistical control problem generally has the following components: (1) a model positing a relationship among output and input variables, (2) input variables which can be controlled, (3) a loss function representing an optimization desideratum by imposing some penalty when output departs from its target.

In problems of control, we can sequentially update the available information using Bayes' theorem according to model assumptions, determine the target output value or interval for the next period, choose a loss function which imposes some penalty when the output departs from its target, and select the design level of the input covariates such that the expected loss function evaluated by the predictive density will be minimized. Learning and design considerations are also involved in the multi-period control problem, because as time proceeds more sample information is available in learning more about the unknown parameters, but how much we learn will depend on the setting of control variables and that is a design problem. By admitting learning into the control problem, at the i^{th} future period, all previous information will be employed to obtain the settings of the independent variables. There are two approaches for deriving a sequence of future values of the independent variables. One is sequential updating which derives solutions by forward induction. Another is dynamic programming which derives solutions by backward induction and also takes into consideration the effect of each future setting on future periods. Although the dynamic programming method provides an optimal sequence of solutions, even for a two-period control problem for a simple linear regression model with quadratic loss there is no explicit solution for the 1^{st} future period (Zellner, 1971). Problems in applying dynamic programming are that the horizon of the process may be unknown or too long to avoid cumbersome calculational efforts for a continually moving horizon. Further, the reduction of expected loss may be negligible when compared to forward induction. Therefore here we shall focus on the sequential updating approach.

Let the model of interest have the following functional relation:

$$y_t = h(x_t, \beta) + \epsilon_t, \quad t = 1, \ldots, N, \tag{1.1}$$

where x_t, a $p \times 1$ vector, may contain lagged dependent or independent variables and $\epsilon_t s$ are *i.i.d.* error terms with mean 0 and variance σ^2. If x_t contains uncontrollable variables, we then partition it into two parts $x_{t(1)}$ and $x_{t(2)}$, where $x_{t(1)}$ is a $p_1 \times 1$ vector with $1 \leq p_1 \leq p$ consists of all the controllable independent variables. The regression coefficient vector β is partitioned in the same manner, with β_1 being a $p_1 \times 1$ vector and β_2 being a $(p-p_1) \times 1$ vector. We assume this functional relation will continue indefinitely. Let D_{i-1} be the collection of available information at the end of the $(N+i-1)^{st}$ period, $z_i \equiv y_{N+i}$, $w_i \equiv x_{N+i}$, $w_{i(1)} \equiv x_{N+i(1)}$, $w_{i(2)} \equiv x_{N+i(2)}$, $e_i \equiv \epsilon_{N+i}$, and $L_i(z_i, a_i)$ be the loss function for the i^{th} future period where a_i is its target value or interval. An object for an M-period control is to minimize the expectation of $\sum_{i=1}^{M} L_i(z_i, a_i)$. The approach considered is sequential updating, which selects $w_{i(1)}$ such that the predictive expected loss (p. e. l.)

$$E[L_i(z_i, a_i)|D_{i-1}] = \int L_i(z_i, a_i) p(z_i|D_{i-1}, w_i)\, dz_i \tag{1.2}$$

is minimized for $i = 1, \ldots, M$, where $p(z_i|D_{i-1}, w_i)$ is the predictive density obtained in a Bayesian formulation.

If $h(x_t, \beta) = x_t'\beta$, and $L(z, a) = (z-a)^2$, then the sequential updating solution will depend on the evaluation of certain posterior expectations, namely

$$w_{i(1)}^{*} = [E(\beta_1\beta_1'|D_{i-1})]^{-1}[E(\beta_1|D_{i-1})a_i - E(\beta_1\beta_2'|D_{i-1})w_{i(2)}]. \tag{1.3}$$

However, the predictive density or the marginal posterior densities may not be mathematically tractable, therefore we may have to employ appropriate sampling techniques to evaluate the posterior moments or the p. e. l. In Section 2, we propose a general Monte Carlo minimum p. e. l. approach which can be applied to a wide range of loss functions. In Section 3, we show that the Monte Carlo minimum p. e. l. can be applied specifically to regression models with nonexplosive autoregressive features. If an explicit and tractable form for the sequential updating rule doesn't exist, then a numerical search is needed. Section 4 provides such a numerical illustration.

2 Monte Carlo Minimum Predictive Expected Loss

By Monte Carlo minimization we mean a minimization procedure in which the objective function cannot be calculated exactly and is approximated by Monte Carlo. The term could also be used to refer to a random search algorithm, such as simulated annealing, but we do not use it that way. This method has been used (Geyer and Thompson, 1992) for maximum likelihood estimation for complex stochastic processes. Here we apply it to minimizing predictive expected loss. The theory developed in this section is similar to that for Monte Carlo maximum likelihood (Geyer, 1994).

When the predictive distribution is analytically intractable, and there is no scheme for generating independent samples from it, expectations with respect to the predictive distribution can still be approximated using the Gibbs sampler or other Markov chain Monte Carlo schemes. It suffices to show how sequential updating works for a single time period, say the $(N+i)^{th}$. Let $(\beta^{(j)}, \sigma^{2(j)})$, $j = 1, 2, \ldots$ be an ergodic Markov chain whose stationary distribution is the ~~predictive~~ posterior distribution for (β, σ^2), and let $e^{(j)}$ be independent mean zero, variance $\sigma^{2(j)}$ random variates. Assume $w_{i(2)}$ is known and the control setting $w_{i(1)}$ is fixed at w. Then

$$z^{(j)}(w) = h\left((w, w_{i(2)}), \beta^{(j)}\right) + e^{(j)}$$

is a functional of the Markov chain whose stationary distribution is the distribution of z_i for the control setting w. Define the p. e. l. at this control setting as

$$r(w) = E[L_i(z_i, a_i)|D_{i-1}]|_{w_{i(1)}=w}.$$

Then its Monte Carlo approximation is

$$r_m(w) = \frac{1}{m}\sum_{j=1}^{m} L_i\left(z^{(j)}(w), a_i\right) \tag{2.1}$$

The w that minimizes $r_m(w)$ approximates the minimum p. e. l. control setting.

Ergodicity guarantees that $r_m(w)$ converges to $r(w)$ for a fixed control setting w for almost all sample paths of the Markov chain Monte Carlo. Ergodicity alone does not guarantee that a minimizer of r_m (call it $\hat{w}_m$) converges to a minimizer of r (call it w^*), even if the minimizer of r is unique.

Before proceeding with the relevant convergence theory, we first generalize the problem to obtain a theory with wider applications. Suppose $L(w, x)$ is a loss function for some decision problem, where w, the action, takes values in some separable metric space $\mathcal{W}$, and x is a random element of some probability space $(\mathcal{X}, \mathcal{B}, P)$. Suppose that $x_1, x_2, \ldots$ is an ergodic Markov chain having stationary distribution P. Let

$$r(w) = EL(w, X) = \int L(w, x)\, dP(x) \tag{2.2}$$

and

$$r_m(w) = \frac{1}{m} \sum_{i=1}^{m} L(w, x_i). \tag{2.3}$$

We wish to study the convergence of r_m to r. The control problem described above is a special case of this general set-up: r and r_m are the same, and the relation between the loss functions is given by

$$L(w, x_j) = L_i\left(h\left((w, w_{i(2)}), \beta^{(j)}\right) + e^{(j)}, a_i\right)$$

where

$$x_j = \left(\beta^{(j)}, e^{(j)}\right).$$

We first show that r_m *epiconverges* to r. Epiconvergence is a type of convergence useful in optimization problems, see Attouch (1984) for an authoritative treatment and Kall (1986) or Geyer (1994) for a brief introduction. The usefulness of epiconvergence comes from the following property, which is Theorem 1.10 in Attouch (1984). Suppose g_n is a sequence of functions epiconverging to a function g, which is denoted $g_n \xrightarrow{e} g$. Suppose that x_n is a sequence of "ϵ-minimizers" of g_n, i. e. that

$$g_n(x_n) \leq \inf_x g_n(x) + \epsilon_n$$

where $\epsilon_n \to 0$. Then every cluster point of the sequence x_n minimizes g, i. e. $x_{n_k} \to x$ implies $g(x) = \inf g$. Moreover the optimal values also converge, i. e. $g_n(x_{n_k}) \to g(x)$.

Theorem 1 *Suppose $L(w,x)$ is a nonnegative function on $\mathcal{W}\times\mathcal{X}$, where $\mathcal{W}$ is a separable metric space and $(\mathcal{X},\mathcal{B},P)$ a probability space, and let $r(w)$ and $r_m(w)$ be defined by (2.2) and (2.3) where $x_1, x_2, \ldots$ is a realization of an ergodic Markov chain having stationary distribution P. Suppose that for all $w \in \mathcal{W}$ the function $L(\cdot,x)$ is lower semicontinuous at w for P-almost all x, where the exception set may depend on w, and is upper semicontinuous at w for P-almost all x, where the exception set does not depend on w. Then $r_m \xrightarrow{e} r$ for almost all sample paths of the Monte Carlo.*

Proof. Since $\mathcal{W}$ is a separable metric space, there is a countable basis $\mathcal{A}_c = \{A_1, A_2, \ldots\}$ of open sets for the topology of $\mathcal{W}$ (i. e. every open set is a union of sets in $\mathcal{A}_c$). Let $\mathcal{N}(w)$ denote the set of neighborhoods of the point w, and let $\mathcal{N}_c(w) = \mathcal{A}_c \cap \mathcal{N}(w)$ be the neighborhoods in the countable basis. Define w_n to be a point of A_n satisfying

$$r(w_n) \le \inf_{u\in A_n} r(u) + \frac{1}{n}, \qquad n = 1,2,\ldots \tag{2.4}$$

and define $\mathcal{W}_c = \{w_1, w_2, \ldots\}$. A countable union of null sets being a null set, we may assume that

$$r_m(w) \to r(w), \qquad \forall w \in \mathcal{W}_c \tag{2.5}$$

and

$$\frac{1}{m}\sum_{i=1}^{m} \inf_{u\in A} L(u,x_i) \to E \inf_{u\in A} L(u,X), \qquad \forall A \in \mathcal{A}_c \tag{2.6}$$

hold simultaneously for almost all sample paths of the Monte Carlo. To establish epiconvergence we need to show that (2.5) and (2.6) imply

$$r(w) \ge \sup_{A\in\mathcal{N}_c(w)} \limsup_{m\to\infty} \inf_{u\in A} r_m(u) \tag{2.7}$$

and

$$r(w) \le \sup_{A\in\mathcal{N}_c(w)} \liminf_{m\to\infty} \inf_{u\in A} r_m(u) \tag{2.8}$$

(Attouch, 1984, pp. 25–26).

We start with (2.7). By (2.5) for all $A \in \mathcal{A}_c$ and all $w \in A \cap \mathcal{W}_c$

$$r(w) = \lim_{m\to\infty} r_m(w) \ge \limsup_{m\to\infty} \inf_{u\in A} r_m(u)$$

$$\inf_{u\in A\cap W_c} r(u) \geq \limsup_{m\to\infty} \inf_{u\in A} r_m(u)$$

and

$$\sup_{A\in\mathcal{N}_c(w)} \inf_{u\in A\cap W_c} r(u) \geq \sup_{A\in\mathcal{N}_c(w)} \limsup_{m\to\infty} \inf_{u\in A} r_m(u) \tag{2.9}$$

The function r is lower semicontinuous, because if $w_n \to w$, then

$$r(w) = EL(w,X) \leq E\liminf_n L(w_n,X) \leq \liminf_n EL(w_n,X) = \liminf_n r(w_n),$$

where the first inequality is lower semicontinuity of L and the second is Fatou's lemma. If $A \in \mathcal{A}_c$ and $w \in A$, any ball B centered at w is contained in A if its radius is small enough, since A is open. Then, since $\mathcal{A}_c$ is a basis there is an $A' \in \mathcal{A}_c$ such that $w \in A' \subset B$. Hence is possible to choose a subsequence n_k such that $A_{n_k} \supset A_{n_{k+1}}$ and $\bigcap_k A_{n_k} = \{w\}$. By lower semicontinuity of $r(w)$

$$\sup_{A\in\mathcal{N}_c(w)} \inf_{u\in A} r(u) = \lim_{k\to\infty} \inf_{u\in A_{n_k}} r(u) = r(w) \tag{2.10}$$

and by (2.4)

$$\inf_{u\in A_{n_k}} r(u) \leq \inf_{u\in A_{n_k}\cap W_c} r(u) \leq r(w_{n_k}) \leq \inf_{u\in A_{n_k}} r(u) + \frac{1}{n_k} \tag{2.11}$$

and (2.9), (2.10) and (2.11) imply (2.7).

Next we prove (2.8). Using the same subsequence n_k as above

$$\begin{aligned}
\sup_{A\in\mathcal{N}_c(w)} \liminf_{m\to\infty} \inf_{u\in A} r_m(u) &\geq \sup_{A\in\mathcal{N}_c(w)} \liminf_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m} \inf_{u\in A} L(u,x_i) \\
&= \sup_{A\in\mathcal{N}_c(w)} E \inf_{u\in A} L(u,X) \\
&= \lim_{k\to\infty} E \inf_{u\in A_{n_k}} L(u,X) \\
&= E \lim_{k\to\infty} \inf_{u\in A_{n_k}} L(u,X) \\
&\geq EL(w,X) \\
&= r(w)
\end{aligned}$$

where the first inequality is subadditivity of the infimum, the first equality is (2.6), the interchange of the limit and expectation is dominated convergence, and the second inequality is lower semicontinuity of L. The upper semicontinuity hypothesis assures that $\inf_{u\in A} L(u,X)$ is measurable. □

In many cases, the loss function $L(w, X)$ is continuous in w for each X, so the upper and lower semicontinuity assumptions are trivially satisfied. One case where the loss function is not continuous, and the upper and lower semicontinuity assumptions are actually needed is the case of a zero-one loss function. For example, suppose that $f(X)$ is a scalar function and the loss function is of the form

$$L(w, x) = \begin{cases} 0, & |w - f(x)| < \epsilon \\ 1, & |w - f(x)| \geq \epsilon \end{cases} \tag{2.12}$$

This function is upper semicontinuous for each x (no exception set required), and for a fixed w is lower semicontinuous at w for all x except for x such that $f(x) = w \pm \epsilon$. Typically, the set of such points will have probability zero, and the condition of the theorem will be satisfied.

Epiconvergence does not, by itself, imply convergence. Some assumption that bars "escape to infinity" is required. The two most common are compactness or convexity.

Theorem 2 *Suppose $r_m \xrightarrow{e} r$ and $\hat{w}_m$ satisfies*

$$r_m(\hat{w}_m) \leq \inf_{w \in \mathcal{W}} r_m(w) + \epsilon_m \tag{2.13}$$

where $\epsilon_m \to 0$. Suppose that $\mathcal{W}$ is compact. Then

1. *Every subsequence of $\{\hat{w}_m\}$ has a convergent subsubseqence $\hat{w}_{m_k}$ whose limit minimizes r. Moreover $r_m(\hat{w}_{m_k}) \to \inf_{w \in \mathcal{W}} r(w)$.*

2. *If r has a unique minimizer w^*, then $\hat{w}_m \to w^*$, and $r_m(\hat{w}_m) \to r(w^*)$.*

Proof. This is just Theorem 1.10 in Attouch (1984) and compactness (see the remarks preceding Theorem 1 above). □

The compactness of the action space must often be manufactured by compactification. In the example with the zero-one loss function, $\mathcal{W}$ is the whole real line, which is not compact. However, we can compactify $\mathcal{W}$ by attaching an ideal point ∞ and taking complements of compact sets as neighborhoods of ∞ (the one-point compactification). The loss function (2.12) is continuous at ∞ if we define $L(\infty, x) = 1$ for all x. Now Theorem 2 applies. Note that ∞ cannot be a cluster point of the minimizing sequence, since $r(\infty) = 1$ cannot be the minimum.

There is one case in which convergence can be obtained without compactification. Suppose that $\mathcal{W}$ is a finite-dimensional Euclidean space and $L(w,x)$ is convex in w for each x (hence continuous, hence satisfying the assumptions for Theorem 1). Then r is also convex. Even if $L(w,x)$ is everywhere finite, $r(w) = EL(w,X)$ need not be finite. Where $L(w,\cdot)$ is not integrable, $r(w) = +\infty$, since $L(w,x)$ is nonnegative. Suppose that the so-called *effective domain* of r, the set

$$\operatorname{dom} r = \{\, w \in \mathcal{W} : r(w) < +\infty \,\}$$

has full dimension (i. e. has a nonempty interior).

Theorem 3 *Suppose the loss is convex, and the effective domain of r has full dimension. Suppose that r has a unique minimizer w^* and $\hat{w}_m$ satisfies (2.13). Then $\hat{w}_m \to w^*$, and $r_m(\hat{w}_m) \to r(w^*)$.*

Proof. Fix a closed ball B centered at w^* and contained in $\operatorname{dom} r$. By Theorem 10.8 of Rockafellar (1970), the convergence of r_m to r is actually uniform on B. By the uniqueness of the minimizer, the minimum of r on the boundary of B is greater than $r(w^*)$, say $r(w^*) + \delta$. Thus there is an m_0 such that for all $m > m_0$ we have $r_m(w^*) < r(w^*) + \delta/3$ and $r_m(w) \geq r(w^*) + 2\delta/3$ for all w on the boundary of B and also $\epsilon_m < \delta/3$. By convexity we must have $r_m(w) \geq r(w^*) + 2\delta/3$ for all w in the exterior of B. Hence $\hat{w}_m$ must lie in B. Now the convergence follows by the same argument as in Theorem 2. □

For differentiable loss functions, it is usually possible to go a step further and obtain a central limit theorem. The proof is very similar to proofs of the asymptotic normality of maximum likelihood estimates and is omitted.

Theorem 4 *Suppose that the Markov chain is ergodic, the minimum expected loss action w^* is unique, and the action space $\mathcal{W}$ is Euclidean and contains an open neighborhood of w^*. Let r and r_m be given by (2.2) and (2.3) and let $\hat{w}_m$ satisfy (2.13). Suppose also that all of the following hold.*

1. *$\hat{w}_m \to w^*$ in probability.*

2. *$r(w) = EL(w,X)$ can be differentiated twice w. r. t. w under the expectation sign.*

3. $B = \nabla^2 r(w^*)$ *is positive definite.*

4. $\sqrt{m}\nabla r_m(w^*) \xrightarrow{\mathcal{D}} N(0, V)$ *for some covariance matrix* V.

5. $\nabla^3 r_m(w)$ *is almost surely bounded uniformly in a neighborhood of* w^*.

Then $\nabla^2 r_m(\hat{w}_m) \to B$ *almost surely, and* $\sqrt{m}(\hat{w}_m - w^*) \xrightarrow{\mathcal{D}} N(0, B^{-1}VB^{-1})$.

Most of the conditions are straightforward and analogous to the asymptotics of maximum likelihood. Condition 1 is implied by Theorem 2 or 3. The usual method of verifying condition 5 by finding a dominating function and applying dominated convergence, also applies to Markov chain Monte Carlo. The matrix B can be estimated by $\nabla^2 r_m(\hat{w}_m)$. The only unusual condition is number 4. This is a Markov chain Central Limit Theorem (CLT), and methods of establishing whether it holds are still the subject of active research. See, for example, Geyer (1992), Chan (1993), Tierney (1994), and Chan and Geyer (1994). When the CLT holds, the asymptotic covariance matrix V is typically a sum of autocovariances and can be estimated by standard time-series methods Hastings (1970), Geyer (1992). For simplicity, we explain the case of a single action variable (so V is a scalar). Define $g(x) = \nabla L(w^*, x)$. Then $\nabla r_m(w^*)$ is the sample average of the time series $g(x_i)$, and V has the form $V = \sum_{t=-\infty}^{+\infty} \gamma_t$ where

$$\gamma_t = \gamma_{-t} = \operatorname{Cov}\Big(g(x_i), g(x_{i+t})\Big) \tag{2.14}$$

is the lag t autocovariance for the stationary time series produced by starting the Markov chain in the stationary distribution (typically the asymptotics do not depend on the initial distribution so there is no loss of generality in assuming stationarity). For multiple action variables, the function g is vector-valued, and (2.14) still holds if "Cov" is interpreted as indicating the covariance matrix of vector valued variables. Then both γ_t and V become matrices.

3 Examples

3.1 Linear Regression Model With Lagged Dependent Variables

The linear regression model with autoregressive responses, also called ARX (autoregressive with covariates) model in econometrics, observed up to the N^{th} period, has the form

$$y_t = y_t^{(q)'}\theta + x_t'\beta + \epsilon_t, \quad t = 1, \ldots, N. \tag{3.1}$$

This model includes both an autoregression component $y_t^{(q)}$ and a regression component x_t, where y_t is the observed random response at period t, $y_t^{(q)} = (y_{t-1}, \ldots, y_{t-q})'$ is a $q \times 1$ vector whose j^{th} element is obtained by taking the j^{th} lag of y_t, $\theta = (\theta_1, \ldots, \theta_q)'$ is a $q \times 1$ autoregression coefficient vector, x_t is the observed explanatory variable at period t, β is a $p \times 1$ regression coefficient vector, and ϵ_t, the error term of the t^{th} period, is *i.i.d.* $N(0, \sigma^2)$. Note that in this setting $y_1^{(q)}$ appears in the 1^{st} period, while it is common that $y_1^{(q)}$ is known before the process starts, therefore we consider $y_1^{(q)}$ as a given $q \times 1$ vector. For convenience, assume all independent variables are controllable. It is reasonable to put restrictions on θ to avoid the model being explosive. The stationary condition for θ is that all the roots of $1 - \sum_1^q \theta_i B^i = 0$ exceed 1 in absolute value (see Box and Jenkins, 1976, p79). We shall assume that θ is independent of β and σ^2 a priori and is uniform over the region where stationarity holds. Let R_q be the region of θ satisfying the stationary condition, *i.e.*

$$R_q = \{(\theta_1, \ldots, \theta_q) : \text{absolute values of roots of } 1 - \sum_1^q \theta_i \mathrm{B}^i = 0 \text{ exceeds } 1\}. \tag{3.2}$$

We further assume that $\pi(\beta|\sigma^{-2})$ is $N(\mu, \sigma^2\tau^{-1})$ and $\pi(\sigma^{-2})$ is $Gamma(\alpha, \gamma)$.

Define $Y_j = (y_1, \ldots, y_{N+j})'$, $Y_j^{(q)} = (y_1^{(q)}, \ldots, y_{N+j}^{(q)})'$, $X_j = (x_1, \ldots, x_{N+j})'$. The updating equations at the beginning of the i^{th} future period are:

$$\begin{aligned}
X_{i-1} &= (X_0', W_{i-1}')' = (X_{i-2}', w_{i-1})' \\
Y_{i-1} &= (Y_0', Z_{i-1}')' = (Y_{i-2}', z_{i-1})' \\
Y_{i-1}^{(q)} &= (Y_0^{(q)'}, Z_{i-1}^{(q)'})' = (Y_{i-2}^{(q)'}, z_{i-1}^{(q)})' \\
A_{i-1} &= X_{i-1}'X_{i-1} + \tau = A_{i-2} + w_{i-1}w_{i-1}',
\end{aligned}$$

$$
\begin{aligned}
B_{i-1} &= X'_{i-1}Y^{(q)}_{i-1} = B_{i-2} + w_{i-1}z^{(q)\,\prime}_{i-1},\\
C_{i-1} &= X'_{i-1}Y_{i-1} + \tau\mu = C_{i-2} + w_{i-1}z_{i-1},\\
H_{i-1} &= Y^{(q)\prime}_{i-1}Y_{i-1} = H_{i-2} + z^{(q)}_{i-1}z_{i-1},\\
E_{i-1} &= Y^{(q)\prime}_{i-1}Y^{(q)}_{i-1} - B'_{i-1}A^{-1}_{i-1}B_{i-1}\\
&= E_{i-2} + (z^{(q)}_{i-1} - \eta'_{i-2}w_{i-1})(1 + w'_{i-1}A^{-1}_{i-2}w_{i-1})^{-1}(z^{(q)\,\prime}_{i-1} - w'_{i-1}\eta_{i-2})\\
F_{i-1} &= H_{i-1} - B'_{i-1}A^{-1}_{i-1}C_{i-1}\\
&= F_{i-2} + (z^{(q)}_{i-1} - \eta'_{i-2}w_{i-1})(1 + w'_{i-1}A^{-1}_{i-2}w_{i-1})^{-1}(z_{i-1} - w'_{i-1}\zeta_{i-2}),\\
Q_{i-1} &= Y'_0Y_0 + 2\gamma + \mu'\tau\mu - C'_{i-1}A^{-1}_{i-1}C_{i-1} - F'_{i-1}E^{-1}_{i-1}F_{i-1}\\
&= Q_{i-2} + [(z_{i-1} - w'_{i-1}\zeta_{i-2}) - (z^{(q)\,\prime}_{i-1} - w'_{i-1}\eta_{i-2})E^{-1}_{i-2}F_{i-2}]^2\\
&\quad \times [1 + w'_{i-1}A^{-1}_{i-2}w_{i-1} + (z^{(q)\,\prime}_{i-1} - w'_{i-1}\eta_{i-2})E^{-1}_{i-2}(z^{(q)}_{i-1} - \eta'_{i-2}w_{i-1})]^{-1},\\
\eta_{i-1} &= A^{-1}_{i-1}B_{i-1},\\
\zeta_{i-1} &= A^{-1}_{i-1}C_{i-1},\\
G_{1,i-1} &= (Y_{i-1} - Y^{(q)}_{i-1}\theta - X_{i-1}\beta)'(Y_{i-1} - Y^{(q)}_{i-1}\theta - X_{i-1}\beta) + 2\gamma + (\beta-\mu)'\tau(\beta-\mu),\\
G_{2,i-1} &= 2\gamma + \mu'\tau\mu + (Y_{i-1} - Y^{(q)}_{i-1}\theta)'(Y_{i-1} - Y^{(q)}_{i-1}\theta)\\
&\quad - (C_{i-1} - B_{i-1}\theta)'A^{-1}_{i-1}(C_{i-1} - B_{i-1}\theta),\\
G_{3,i-1} &= (Y_{i-1} - X_{i-1}\beta)'Q_{Y^{(q)}_{i-1}}(Y_{i-1} - X_{i-1}\beta) + (\beta-\mu)'\tau(\beta-\mu) + 2\gamma.
\end{aligned}
$$

where $i = 2, \ldots$, and at the end of the N^{th} period

$$
\begin{aligned}
A_0 &= \tau + X'_0X_0, & B_0 &= X'_0Y^{(q)}_0,\\
C_0 &= \tau\mu + X'_0Y_0, & H_0 &= Y^{(q)\prime}_0Y_0,\\
E_0 &= Y^{(q)\prime}_0Y^{(q)}_0 - B'_0A^{-1}_0B_0, & F_0 &= H_0 - B'_0A^{-1}_0C_0,
\end{aligned}
$$
$$
Q_0 = Y'_0Y_0 + 2\gamma + \mu'\tau\mu - C'_0A^{-1}_0C_0 - F'_0E^{-1}_0F_0.
$$

Theorem 5 *Under the above prior assumption, at the beginning of the i^{th} future period, we obtain the following results:*

1. *Conditional posteriors for σ^{-2}:*
   $\pi(\sigma^{-2}|\beta,\theta,D_{i-1})$ *is Gamma* $\left(\frac{N+i-1+2\alpha+p}{2}, \frac{1}{2}G_{1,i-1}\right)$.
   $\pi(\sigma^{-2}|\theta,D_{i-1})$ *is Gamma* $\left(\frac{N+i-1+2\alpha}{2}, \frac{1}{2}G_{2,i-1}\right)$.

2. *Conditional posteriors for β:*

$\pi(\beta|\theta,\sigma^2,D_{i-1})$ *is* $N_p(\zeta_{i-1}-\eta_{i-1}\theta,\sigma^2 A_{i-1}^{-1})$, *a $p-$variate normal density.*

$\pi(\beta|\theta,D_{i-1})$ *is* $t_p\left(N+i+2\alpha-1;\zeta_{i-1}-\eta_{i-1}\theta,\frac{G_{2,i-1}}{N+i+2\alpha-3}A_{i-1}^{-1}\right)$, *where* $t_p(\nu,m,V)$ *is a multivariate student t density with ν d.f., mean m and covariance V.*

3. *Conditional posteriors for θ:*

$\pi(\theta|\beta,\sigma^2,D_{i-1})$ *is a q-variate truncated normal density satisfying* $\theta\in R_q$, *where the untruncated normal has mean* $(Y_{i-1}^{(q)\prime}Y_{i-1}^{(q)})^{-1}Y_{i-1}^{(q)\prime}(Y_{i-1}-X_{i-1}\beta)$, *and covariance* $\sigma^2(Y_{i-1}^{(q)\prime}Y_{i-1}^{(q)})^{-1}$.

$\pi(\theta|\sigma^2,D_{i-1})$ *is a q-variate truncated normal density satisfying* $\theta\in R_q$, *where the untruncated normal is* $N_q(E_{i-1}^{-1}F_{i-1},\sigma^2E_{i-1}^{-1})$.

$\pi(\theta|\beta,D_{i-1})$ *is a q-variate truncated t satisfying* $\theta\in R_q$, *where the untruncated t has* $N+2\alpha+i+p-q-1$ *d.f., mean* $(Y_{i-1}^{(q)\prime}Y_{i-1}^{(q)})^{-1}Y_{i-1}^{(q)\prime}(Y_{i-1}-X_{i-1}\beta)$, *and covariance* $G_{3,i-1}(Y_{i-1}^{(q)\prime}Y_{i-1}^{(q)})^{-1}(N+i+2\alpha+p-q-3)^{-1}$.

4. *Marginal posterior for θ:*

$\pi(\theta|D_{i-1})$ *is a q-variate truncated t satisfying* $\theta\in R_q$, *where the untruncated t is* $t\left(N+i+2\alpha-q-1;E_{i-1}^{-1}F_{i-1},\frac{Q_{i-1}E_{i-1}^{-1}}{N+i+2\alpha-q-3}\right)$.

5. *Conditional predictive distribution:*

$p(z_i|\theta,D_{i-1},w_i)$ *is a univariate t with d.f.* $N+2\alpha+i-1$, *mean* $w_i'\zeta_{i-1}+(z_i^{(q)\prime}-w_i'\eta_{i-1})\theta$, *and variance* $(1+w_i'A_{i-1}^{-1}w_i)\frac{Q_{i-1}+(\theta-E_{i-1}^{-1}F_{i-1})'E_{i-1}(\theta-E_{i-1}^{-1}F_{i-1})}{N+2\alpha+i-3}$.

6. *Predictive distribution:*

$$
\begin{aligned}
p(z_i|D_{i-1},w_i) &= \frac{\Gamma\left(\frac{N+i+2\alpha}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{N+i-1+2\alpha}{2}\right)}\left|\frac{A_{i-1}}{A_i}\right|^{\frac{1}{2}}\frac{Q_i^{-\frac{N+i+2\alpha}{2}}}{Q_{i-1}^{-\frac{N+i-1+2\alpha}{2}}} \\
&\times\frac{\int_{R_q}\left(1+\frac{(\theta-E_i^{-1}F_i)'E_i(\theta-E_i^{-1}F_i)}{Q_i}\right)^{-\frac{N+i+2\alpha}{2}}d\theta}{\int_{R_q}\left(1+\frac{(\theta-E_{i-1}^{-1}F_{i-1})'E_{i-1}(\theta-E_{i-1}^{-1}F_{i-1})}{Q_{i-1}}\right)^{-\frac{N+i-1+2\alpha}{2}}d\theta} \\
&= t\Big(N+i-1+2\alpha;\, z_i^{(q)\prime}E_{i-1}^{-1}F_{i-1}+w_i'(\zeta_{i-1}-\eta_{i-1}E_{i-1}^{-1}F_{i-1}), \\
&\qquad \frac{Q_{i-1}[1+w_i'A_{i-1}^{-1}w_i+(z_i^{(q)}-\eta_{i-1}'w_i)'E_{i-1}^{-1}(z_i^{(q)}-\eta_{i-1}'w_i)]}{N+i-3+2\alpha}\Big)
\end{aligned}
$$

$$\times \left|\frac{E_i}{E_{i-1}}\right|^{\frac{1}{2}} \frac{\int_{R_q}\left(1+\frac{(\theta-E_i^{-1}F_i)'E_i(\theta-E_i^{-1}F_i)}{Q_i}\right)^{-\frac{N+i+2\alpha}{2}}d\theta}{\int_{R_q}\left(1+\frac{(\theta-E_{i-1}^{-1}F_{i-1})'E_{i-1}(\theta-E_{i-1}^{-1}F_{i-1})}{Q_{i-1}}\right)^{-\frac{N+i-1+2\alpha}{2}}d\theta}.$$

7. *The optimal control solution under quadratic loss is*

$$\begin{aligned} w_i^* = {} & [((\zeta_{i-1}-\eta_{i-1}E(\theta|D_{i-1}))(\zeta_{i-1}-\eta_{i-1}E(\theta|D_{i-1}))' \\ & + \frac{A_{i-1}^{-1}}{N+2\alpha+i-3}(y^Ty+2\gamma+\mu^T\tau\mu-C_{i-1}'A_{i-1}^{-1}C_{i-1}-E(\theta|D_{i-1})'F_{i-1} \\ & - F_{i-1}'E(\theta|D_{i-1})+E(\theta'E_{i-1}\theta|D_{i-1}))+\eta_{i-1}V(\theta|D_{i-1})\eta_{i-1}']^{-1} \\ & \times [((\zeta_{i-1}-\eta_{i-1}E(\theta|D_{i-1}))a_i-(\zeta_{i-1}E(\theta|D_{i-1})'-\eta_{i-1}E(\theta\theta'|D_{i-1}))z_i^{(q)}]. \end{aligned}$$

Proof. The conditional posterior probability density of σ^2 and β and the posterior probability density of θ follow directly from Bayes' theorem. To find the optimal solution, we first derive the conditional posterior moments and then the posterior moments. The conditional posterior moments are easily derived from their conditional posterior probability densities. To derive the conditional predictive density, not only Bayes' theorem, but also the updating equations for A_i, E_i, and F_i are useful, because the conditional predictive density is

$$\begin{aligned} p(z_i|\theta, D_{i-1}, w_i) \propto {} & [(Y_{i-1}-Y_{i-1}^{(q)'}\theta)'(Y_{i-1}-Y_{i-1}^{(q)'}\theta)+(z_i-z_i^{(q)'}\theta)^2 \\ & -(C_i-B_i\theta)'A_i^{-1}(C_i-B_i\theta)+2\gamma+\mu'\tau\mu]^{-\frac{N+2\alpha+i}{2}}. \quad \square \end{aligned}$$

From Theorem 5, it is readily observed that the predictive density is intractable but all full conditional posterior densities are tractable. Therefore, to evaluate the p. e. l. at a fixed input setting, one may consider generating the response for the next period corresponding to that input setting by the Gibbs sampler. Let $(\theta^{(j)}, \beta^{(j)}, \sigma^{-2\,(j)}, e^{(j)})$, $j = 1, \ldots, m$, be a Markov chain generated by the Gibbs sampler, where $e^{(j)}$ is from $N(0, \sigma^{2\,(j)})$. To claim that r_m, defined by (2.1), be the Monte Carlo approximant of the p. e. l. we need to verify the ergodicity of the Markov chain simulated by Gibbs sampling. Because the posterior distributions of θ, β and σ^{-2} conditional on D_{i-1} don't depend on ϵ_{N+i}, it suffices to show that the sequence $\{\theta^{(j)}, \beta^{(j)}, \sigma^{-2\,(j)}\}$ is ergodic.

Theorem 6 *The Markov chain constructed by the Gibbs sampler, $(\theta^{(j)}, \beta^{(j)}, \sigma^{-2\,(j)})$, $j = 1, 2, \ldots$, is irreducible, aperiodic, and ergodic.*

Proof. All three full conditional distributions $\pi(\theta|\beta, \sigma^{-2}, D_{i-1})$, $\pi(\beta|\theta, \sigma^{-2}, D_{i-1})$, and $\pi(\sigma^{-2}|\theta, \beta, D_{i-1})$ have positive densities over the parameter space, and they are absolutely continuous *w.r.t.* their marginal posterior distributions. Let $\pi(\theta, \beta, \sigma^{-2}|D_{i-1})$ denote the joint posterior distribution, $\phi^{(0)}$ be the initial value of $(\theta, \beta, \sigma^{-2})$, and $P^n(\phi^{(0)}, .)$ denote the n^{th} step transition probability for $(\theta^{(n)}, \beta^{(n)}, \sigma^{-2(n)})$. The proof is by contradiction: suppose the chain is not irreducible, then for some $\phi^{(0)}$ and measurable set A, $\pi(A|D_{i-1}) > 0$, and $P^n(\phi^{(0)}, A) = 0\ \forall\ n$. But for this A and $\phi^{(0)}$, $P^1(\phi^{(0)}, A) = 0$ implies $\pi(A|D_{i-1}) = 0$, because of absolute continuity, therefore this chain must be irreducible. Now assume the chain is periodic with period $d \geq 2$, then there exists a sequence of nonempty disjoint sets $A_0, A_1, \ldots, A_{d-1}$, such that $\pi(A_k|D_{i-1}) > 0, \forall k = 0, \ldots, d-1$, and for all $\phi = (\theta, \beta, \sigma^{-2}) \in A_k$ where $k \in \{0, \ldots, d-1\}$, $P^1(\phi \in A_k, A_j) = 1$, $\forall j = k+1(mod\ d)$. But due to absolute continuity, $\pi(A_j|D_{i-1}) > 0$ implies $P^1(\phi \in A_k, A_j) > 0$, $\forall\ k, j = 0, \ldots, d-1$, which contradicts that the chain is of period d. Because the chain is irreducible, then by Corollary 1 of Tierney (1994), the chain is positive Harris recurrent. Hence it is also ergodic. □

Therefore, we are able to produce an ergodic chain of $(\theta, \beta, \sigma^{-2}, e)$ by Gibbs sampling. If the sufficient conditions of Theorem 1 and Theorem 2 are satisfied, then the Monte Carlo minimum p. e. l. approach provides Monte Carlo approximants for the sequential updating setting and the infimum of the p. e. l.

3.2 Dynamic Input-Output Models

Consider a dynamic input-output system which has lagged components for both output and input of the form

$$y_t = y_t^{(q)'}\theta + x_t\beta_1 + x_{t-1}\beta_2 + \ldots + x_{t-p}\beta_{p-1} + \epsilon_t, \quad t = 1, \ldots \tag{3.3}$$

where θ is a $q \times 1$ autoregressive coefficient, $y_t^{(q)} = (y_{t-1}, \ldots, y_{t-q})'$ is a $q \times 1$ vector, x_t is a scalar, and ϵ_t, the error term of the t^{th} period, is *i.i.d.* $N(0, \sigma^2)$. Define

$x_t^{(p-1)}=(x_{t-1},\ldots,x_{t-p})'$ be a $(p-1)\times 1$ vector, and $u_t=(x_t,x_t^{(p-1)})'$ be a $p\times 1$ vector, then (3.3) can also be written as

$$y_t = y_t^{(q)'}\theta + u_t'\beta + \epsilon_t. \tag{3.4}$$

We assume $y_1^{(q)}$ and $x_1^{(p-1)}$ are both known at period 1. Define $Y_j=(y_1,\ldots,y_{N+j})'$, $Y_j^{(q)}=(y_1^{(q)},\ldots,y_{N+j}^{(q)})'$, $U_j=(X_j,X_j^{(p-1)})$, $X_j=(x_1,\ldots,x_{N+j})'$, and $X_j^{(p-1)}=(x_1^{(p-1)},\ldots,x_{N+j}^{(p-1)})'$.

The requirement for this model being nonexplosive exactly parallels that of stationarity for the ARMA models (Box and Jenkins, 1976, p346). Define $\theta(t)=1-\theta_1 t-\ldots-\theta_q t^q$, then for this model to be nonexplosive θ is restricted to R_q, where R_q is defined by (3.2). We shall assume θ is independent of β and σ^2 a priori and is uniform over the region where the nonexplosive condition holds. Replace X_{i-1} by U_{i-1} into A_{i-1}, B_{i-1}, C_{i-1}, E_{i-1}, F_{i-1}, Q_{i-1}, $G_{1,i-1}$, $G_{2,i-1}$,$G_{3,i-1}$, ζ_{i-1}, and η_{i-1} defined in Section 3.1, then we establish the following theorem.

Theorem 7 *Let $\beta|\sigma^2$ be $N(\mu,\sigma^2\tau^{-1})$, and σ^{-2} be Gamma(α,γ) a priori.*

1. *The following posterior distributions: $\pi(\sigma^{-2}|\beta,\theta,D_{i-1})$, $\pi(\sigma^{-2}|\beta,D_{i-1})$, $\pi(\beta|\theta,\sigma^{-2},D_{i-1})$, $\pi(\beta|\theta,D_{i-1})$, $\pi(\theta|\beta,\sigma^2,D_{i-1})$, $\pi(\theta|\sigma^2,D_{i-1})$, $\pi(\theta|\beta,D_{i-1})$, and $\pi(\theta|D_{i-1})$ have the same forms as those in Theorem 5.*

2. *The conditional predictive distribution $p(y_{N+i}|\theta,D_{i-1},u_{N+i})$ and the marginal predictive distribution $p(y_{N+i}|D_{i-1},u_{N+i})$ have the same form as those in Theorem 5.*

3. *Under quadratic loss, let a be the target for the $(N+i)^{st}$ period, then the p. e. l., $r(x_{N+i})$, is strictly convex and the optimal sequential updating rule is of the form*

$$x_{N+i}^* = \frac{E(\beta_1|D_{i-1})a - E(\beta_1\theta'|D_{i-1})y_{N+i}^{(q)} - E(\beta_1\vec{\beta_2}'|D_0)x_{N+1}^{(p-1)}}{E(\beta_1^2|D_{i-1})},$$

where $\vec{\beta_2}=(\beta_2,\ldots,\beta_p)'$.

Because nonexplosive dynamic input-output models have the same types of conditional and marginal posterior densities as in section (3.1), the ergodicity of the Markov chain simulated by Gibbs sampling follows from the same argument as in Theorem 6.

3.3 Linear Regression Models With Autocorrelated Errors

The linear regression model with autocorrelated errors has the form

$$\begin{aligned} y_t &= x_t'\beta + u_t, \\ u_t &= \theta_1 u_{t-1} + \ldots + \theta_q u_{t-q} + \epsilon_t, \end{aligned} \tag{3.5}$$

where $\epsilon_t' s$ are *i.i.d.* $N(0, \sigma^2)$, and x_t and β are $p \times 1$ vectors. This model can also be written as

$$\theta(B)y_t \;=\; \theta(B)x_t'\beta + \epsilon_t, \tag{3.6}$$

where $\theta(B) = 1 - \theta_1 B - \ldots - \theta_q B^q$, and B is a backshift operator such that $B^j y_t = y_{t-j}$. Define $x_t(\theta)=\theta(B)x_t$, $X_j = (x_1, \ldots, x_{N+j})'$, $X_j(\theta)=(x_1(\theta), \ldots, x_{N+j}(\theta))'$, $y_t(\theta)=\theta(B)y_t$, $Y_j = (y_1, \ldots, y_{N+j})$, $Y_j(\theta) = \theta(B)Y_j$, $U_j = (u_1, \ldots, u_{N+j})'$, $u_t^{(q)} = (u_{t-1}, \ldots, u_{t-q})'$, and $U_j^{(q)} = (u_1^{(q)}, \ldots, u_{N+j}^{(q)})'$. When some of the independent variables are not controllable, we partition x_t, $x_t(\theta)$, and β into $x_{t(1)}$, $x_{t(2)}$, $x_{t(1)}(\theta)$, $x_{t(2)}(\theta)$, and $\beta_{(1)}$, $\beta_{(2)}$, and the symbol with subscript (1) denotes the subvector with the p_1 controllable independent variables, where $1 \le p_1 \le p$.

For the model to be nonexplosive, then the constraint is $\theta \in R_q$ as defined in (3.2). Since $y_0, y_{-1}, \ldots, y_{-q+1}$, $x_0, x_{-1}, \ldots, x_{-q+1}$ appear in the model at the 1^{st} period, they are assumed to be known. Let D_{i-1} denote all information available at the end of the $(N+i-1)^{st}$ period. We further assume that θ is independent of (β, σ^2) a priori, and the prior for (β, σ^2) is $\pi(\beta|\sigma^2)$ is $N(\mu, \sigma^2\tau^{-1})$ and $\pi(\sigma^{-2})$ is $Gamma(\alpha, \gamma)$.

The regression model with autoregressive errors is a widely applied model in a variety of areas. Chib (1993) discussed Bayesian inference for this model with both normal and student t errors and showed how the Gibbs sample can be used for parameter estimation and prediction. Here we demonstrate the Bayesian analysis for this model for purposes of sequential control. To derive the posterior densities and the predictive density, we further define

$$\begin{aligned} A_{i-1}(\theta) &= \tau + X_{i-1}(\theta)'X_{i-1}(\theta), \\ C_{i-1}(\theta) &= \tau\mu + X_{i-1}(\theta)'Y_{i-1}(\theta), \end{aligned}$$

$$Q_{i-1}(\theta) = Y_{i-1}(\theta)'Y_{i-1}(\theta) + 2\gamma + \mu'\tau\mu - C_{i-1}(\theta)'A_{i-1}(\theta)^{-1}C_{i-1}(\theta),$$
$$G_{i-1}(\theta) = 2\gamma + (\beta-\mu)'\tau(\beta-\mu) + U_{i-1}'Q_{U_{i-1}^{(q)}}U_{i-1},$$
$$H_{i-1}(\theta) = 2\gamma + (\beta-\mu)'\tau(\beta-\mu) + (Y_{i-1}(\theta) - X_{i-1}(\theta)\beta)'(Y_{i-1}(\theta) - X_{i-1}(\theta)\beta),$$

where $Q_U = I - U(U'U)^{-1}U'$, and I is the identity matrix.

Theorem 8 *Under the above prior assumption for $(\theta, \beta, \sigma^{-2})$, we obtain:*

1. *Conditional posteriors for σ^{-2}:*
   $\pi(\sigma^{-2}|\beta,\theta,D_{i-1})$ *is Gamma* $\left(\frac{N+i-1+2\alpha+p}{2}, \frac{1}{2}H_{i-1}(\theta)\right)$.
   $\pi(\sigma^{-2}|\theta,D_{i-1})$ *is Gamma* $\left(\frac{N+i-1+2\alpha}{2}, \frac{1}{2}Q_{i-1}(\theta)\right)$.

2. *Conditional posteriors for β:*
   $\pi(\beta|\theta,\sigma^2,D_{i-1})$ *is* $N_p(A_{i-1}(\theta)^{-1}C_{i-1}(\theta), \sigma^2 A_{i-1}(\theta)^{-1})$.
   $\pi(\beta|\theta,D_{i-1})$ *is a p-variate t with* $N+i-1+2\alpha$ *d.f., covariance* $\frac{Q_{i-1}(\theta)}{N+i+2\alpha-3}A_{i-1}(\theta)^{-1}$, *and mean* $A_{i-1}(\theta)^{-1}C_{i-1}(\theta)$.

3. *Conditional posteriors for θ:*
   $\pi(\theta|\beta,\sigma^2,D_{i-1})$ *is a q-variate truncated normal satisfying* $\theta \in R_q$, *where the untruncated normal is* $N_q((U_{i-1}^{(q)\prime}U_{i-1}^{(q)})^{-1}U_{i-1}^{(q)\prime}U_{i-1}, \sigma^2(U_{i-1}^{(q)\prime}U_{i-1}^{(q)})^{-1})$.
   $\pi(\theta|\sigma^2,D_{i-1}) \propto \sigma^{-q}\exp[-\frac{1}{2\sigma^2}Q_{i-1}(\theta)] \times I_{R_q}(\theta)$.
   $$\text{where} \quad I_{R_q}(\theta) = \begin{cases} 1 & \text{if } \theta \in R_q \\ 0 & \text{otherwise,} \end{cases}$$
   $\pi(\theta|\beta,D_{i-1})$ *is a q-variate truncated t satisfying* $\theta \in R_q$, *where the untruncated t is*
   $$t_q\left(N+i-1+2\alpha+p-q; (U_{i-1}^{(q)\prime}U_{i-1}^{(q)})^{-1}U_{i-1}^{(q)\prime}U_{i-1}, \frac{H_{i-1}(\theta)\ \ (U_{i-1}^{(q)\prime}U_{i-1}^{(q)})^{-1}}{N+i+2\alpha+p-q-3}\right).$$

4. *Marginal posterior for θ:*
   $\pi(\theta|D_{i-1}) \propto [\,Q_{i-1}(\theta)\,]^{-\frac{N+i-1+2\alpha}{2}} \times I_{R_q}(\theta)$.

5. *Conditional predictive distribution : $p(y_{N+i}|\theta,D_{i-1},x_{N+i})$ is a univariate t with* $N+i-1+2\alpha$ *d.f., mean* $y_{N+i}^{(q)\prime}\theta + x_{N+i}(\theta)'A_{i-1}(\theta)^{-1}C_{i-1}(\theta)$, *and variance* $\frac{Q_{i-1}(\theta)}{N+i+2\alpha-3}(1 + x_{N+i}(\theta)'A_{i-1}(\theta)^{-1}x_{N+i}(\theta))$.

6. Under quadratic loss, let a be the target for the $(N+i)^{th}$ period, then the p. e. l. $r(x_{N+i})$, is strictly convex and the optimal sequential updating rule is of the form

$$\begin{aligned} x^*_{N+i(1)} = {} & E(\beta_{(1)}\beta'_{(1)}|D_{i-1})^{-1}[\, E(\beta_{(1)}|D_{i-1})a - E(\beta_{(1)}\theta'|D_{i-1})y^{(q)}_{N+i} \\ & - E(\beta_{(1)}\beta'_{(2)}|D_{i-1})x_{N+i(2)} + E(\beta_{(1)}\theta' x^{(q)}_{N+i}\beta|D_{i-1})\,]. \end{aligned}$$

Note that all posterior distributions of θ depend on the error terms $u_t's$, which are generally unobserved, but for given y_t, x_t, β, and σ^2, then $u_t = y_t - x'_t\beta$ becomes degenerate. Again, the ergodicity follows from the same argument as Theorem 6. Therefore, the Gibbs sampling approach can be applied to simulate $(\theta, \beta, \sigma^{-2})$.

4 Numerical Illustration

A set of data was generated, where 20 x's were drawn from $N(2, .25)$, ϵ's were drawn from $N(0, .25)$, and y's were drawn by letting $y_i = .8y_{i-1} + 2x_i + \epsilon_i$, a linear regression model with a lagged dependent variable, with $y_0 = 25$. This data set is listed in Table 7. The target value for next 10 future periods is $a_i = 29$. The prior assumption is that θ is uniform in $(0, 1)$, and $\pi(\beta, \sigma^{-2}) \propto \sigma^2$. At each control period, a total of 100,400 samples were simulated. After we discarded the first 400 samples, we then subsampled 5000 samples with a spacing size 20 to reduce the dependency between samples generated at different iterations. Figure 1 contains scatter plots for θ, β and σ. Figure 2 contains empirical autocovariance plots for θ, β and σ. For related work in the context of the variance and autocovariance of a Markov chain Monte Carlo see Geyer (1992). Figure 3 shows posterior densities for θ, β and σ^{-2}. All dashed lines are density estimates based on full conditional posteriors. The solid line for θ is the exact posterior density, while it is the estimated posterior density based on the reduced conditional posterior for β or σ^{-2}.

(Figures 1–3 about here)

The loss functions being studied are asymmetric quadratic

$$\begin{aligned} L(z_i, 29) &= (z_i - 29)^2 \quad \text{if } z_i \geq 29 \\ &= c_2(z_i - 29)^2 \quad \text{otherwise,} \end{aligned}$$

and asymmetric linear

$$\begin{aligned} L(z_i, 29) &= (z_i - 29) \quad \text{if } z_i \geq 29 \\ &= c_4(29 - z_i) \quad \text{otherwise.} \end{aligned}$$

Let c_2 and c_4 be $0.1, 0.2, 0.5, 1, 2, 5$, and 10 respectively. For every asymmetric quadratic loss, $\hat{w}_i$, a minimizer of the estimated p. e. l. was selected by the Newton-Raphson method; while it was selected by the bisection method for every asymmetric linear loss. The results and data are presented in Tables 1–7.

(Tables 1–7 about here)

Since the regression parameter for x is positive, it is always true that the control setting, corresponding to a loss function that assigns smaller penalty when output is less than the target, is smaller than the control setting, corresponding to the same loss function that assigns greater penalty when output is greater than the target, except for $\hat{w}_2$. As to the realization due to the control procedure, the order of the resulting output is in the same order of c_2 and c_4. That is, smaller c_2 or c_4 not only cause smaller outcomes but tend to select a setting that produces a response less than its target. Also due to the model's positive dependence with regard to previous responses, a larger setting for the next period is chosen whenever the resulting response is smaller than the target.

A robustness discussion is based on the ratio of expected loss associated with using the settings for w from the asymmetric loss to that associated with using the settings for w from the symmetric loss. It is not surprising that as the loss function moves away from symmetry this ratio decreases. This implies that solutions derived from symmetric losses are not robust when there are large differences in the penalties depending on whether the response is less than or greater than the target value.

5 Discussion

Optimal control problems are important in many areas. Here we have presented a methodology for computing sequential updating solutions to such problems using Monte Carlo approximation of the predictive expected loss function. This permits numerical solution of many problems that were previously intractable. As with other problems to which Markov chain Monte Carlo has been applied, the method seems almost unlimited. It can be applied to very complex stochastic models and very complicated loss functions.

The theorems proved in Section 2 actually apply to a broad class of decision problems, of which optimal control is obviously a subset. The theory and methodology resemble those developed for maximum likelihood in complex stochastic processes by Geyer and Thompson (1992) and Geyer (1994), but the theory here is actually simpler because loss functions are nonnegative, whereas log likelihoods are usually unbounded above and below. The method can be applied to any decision problem for which the stochastic part of the model can be simulated by Markov chain Monte Carlo. We hope that these methods will find application to decision problems in many areas.

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Table 1

Sequential Settings under Asymmetric Quadratic Losses

period	$c_2 = 0.1$	$c_2 = 0.2$	$c_2 = 0.5$	$c_2 = 1$	$c_2 = 2$	$c_2 = 5$	$c_2 = 10$
21	3.690064	3.780663	3.907234	4.007551	4.114177	4.260927	4.379241
22	3.033556	3.038224	3.038505	3.035988	3.024827	3.005396	2.984222
23	2.625559	2.640794	2.662917	2.679034	2.699479	2.726488	2.747610
24	2.304994	2.328975	2.362463	2.387885	2.412750	2.443526	2.464335
25	3.079965	3.089390	3.102582	3.111207	3.121058	3.133783	3.143992
26	2.702280	2.715246	2.732186	2.745935	2.758898	2.776886	2.790065
27	2.781206	2.796463	2.816737	2.832459	2.847204	2.866480	2.880429
28	2.947963	2.961764	2.980415	2.990130	3.008944	3.027199	3.041166
29	2.708084	2.723563	2.742154	2.768998	2.768773	2.786941	2.799275
30	3.024678	3.037510	3.057839	3.069437	3.090370	3.110575	3.126137

$L(z_i, 29) = (z_i - 29)^2 I(z_i \geq 29) + c_2(29 - z_i)^2 I(z_i < 29)$

Table 2

Realization of Responses under Asymmetric Quadratic Losses

period	$c_2 = 0.1$	$c_2 = 0.2$	$c_2 = 0.5$	$c_2 = 1$	$c_2 = 2$	$c_2 = 5$	$c_2 = 10$
21	28.10831	28.28951	28.54265	28.74329	28.95654	29.25004	29.48667
22	29.00682	29.16111	29.36419	29.51966	29.66794	29.86388	30.01083
23	29.52383	29.67774	29.88444	30.04106	30.20057	30.41134	30.57115
24	27.91735	28.08843	28.32077	28.49691	28.67425	28.90442	29.07388
25	28.77850	28.93422	29.14647	29.30463	29.46621	29.67579	29.83180
26	28.59420	28.74471	28.94839	29.10241	29.25760	29.46124	29.61241
27	28.24934	28.40026	28.60375	28.75842	28.91205	29.11352	29.26235
28	28.75821	28.90655	29.10665	29.24980	29.41035	29.60803	29.75503
29	28.06787	28.21750	28.41476	28.58297	28.71095	28.90544	29.04770
30	28.14078	28.28615	28.48461	28.64238	28.78663	28.98263	29.12757

$L(a_i, 29) = (z_i - 29)^2 I(z_i \geq 29) + c_2(29 - a_i)^2 I(a_i < 29)$

Table 3

Sequential Settings under Asymmetric Linear Losses

period	$c_4 = 0.1$	$c_4 = 0.2$	$c_4 = 0.5$	$c_4 = 1$	$c_4 = 2$	$c_4 = 5$	$c_4 = 10$
21	3.562614	3.682915	3.862328	4.014915	4.174324	4.417850	4.612614
22	3.032745	3.030501	3.037421	3.029605	3.020526	2.965340	2.910367
23	2.582988	2.609320	2.641278	2.670477	2.697931	2.729352	2.759201
24	2.287291	2.316791	2.362579	2.396527	2.429559	2.480887	2.499319
25	3.058309	3.074969	3.103314	3.114774	3.131842	3.148905	3.173862
26	2.681350	2.693964	2.716619	2.739947	2.769306	2.795404	2.809714
27	2.762057	2.784894	2.806464	2.826849	2.847528	2.877902	2.899350
28	2.927960	2.946198	2.977206	2.994867	3.014886	3.044791	3.063135
29	2.693270	2.713153	2.737054	2.751935	2.777985	2.804309	2.824179
30	3.006889	3.024555	3.057990	3.082220	3.105755	3.134674	3.146723

$L(z_i, 29) = (z_i - 29)I(z_i \geq 29) + c_4(29 - z_i)I(z_i < 29)$

Table 4

Realization of responses under Asymmetric Linear Losses

period	$c_2 = 0.1$	$c_2 = 0.2$	$c_2 = 0.5$	$c_2 = 1$	$c_2 = 2$	$c_2 = 5$	$c_2 = 10$
21	27.85341	28.09402	28.45284	28.75802	29.07683	29.56389	29.95341
22	28.80127	28.98927	29.29017	29.51872	29.75557	30.03484	30.23652
23	29.27426	29.47732	29.78195	30.02319	30.26758	30.55384	30.77488
24	27.68228	27.90373	28.23901	28.49990	28.76148	29.09314	29.30683
25	28.54713	28.75761	29.08253	29.31416	29.55756	29.85701	30.07788
26	28.36725	28.56086	28.86610	29.09806	29.35150	29.64326	29.84857
27	28.02948	28.23004	28.51737	28.74371	28.98782	29.28197	29.48912
28	28.54232	28.73924	29.03113	29.24752	29.48284	29.77798	29.98039
29	27.86552	28.06283	28.34414	28.78738	28.78738	29.07613	29.27780
30	27.94333	28.13650	28.42842	28.87854	28.87854	29.16738	29.35281

$L(z_i, 29) = (z_i - 29)I(z_i \geq 29) + c_4(29 - z_i)I(z_i < 29)$

Table 5

Monte Carlo Expected Asymm. Quadratic Losses of Sequential Settings

period	$c_2 = 0.1$	$c_2 = 0.2$	$c_2 = 0.5$	$c_2 = 1$	$c_2 = 2$	$c_2 = 5$	$c_2 = 10$
21	0.128385	0.203485	0.362683	0.528653	0.806527	1.304448	1.837320
	(0.4891)	(0.6919)	(0.9324)	(1.0000)	(0.9341)	(0.7198)	(0.5414)
22	0.089582	0.136620	0.229007	0.323601	0.456023	0.679020	0.895262
	(0.4913)	(0.6883)	(0.9275)	(1.0000)	(0.9325)	(0.6977)	(0.5028)
23	0.091518	0.141020	0.239215	0.330452	0.483839	0.725518	0.956843
	(0.4771)	(0.6752)	(0.9202)	(1.0000)	(0.9386)	(0.7067)	(0.5094)
24	0.113837	0.173456	0.290459	0.393172	0.574839	0.848495	1.106473
	(0.4733)	(0.6676)	(0.9140)	(1.0000)	(0.9459)	(0.7145)	(0.5137)
25	0.093616	0.144354	0.244374	0.350224	0.490162	0.728692	0.953472
	(0.4983)	(0.7007)	(0.9383)	(1.0000)	(0.9204)	(0.6768)	(0.4807)
26	0.090187	0.137748	0.231561	0.340239	0.461258	0.683696	0.891668
	(0.4871)	(0.6839)	(0.9256)	(1.0000)	(0.9339)	(0.6966)	(0.4971)
27	0.088164	0.134951	0.226976	0.322014	0.451500	0.668049	0.870227
	(0.4877)	(0.6858)	(0.9273)	(1.0000)	(0.9315)	(0.6926)	(0.4932)
28	0.085105	0.130003	0.218428	0.309054	0.435172	0.644766	0.842161
	(0.5073)	(0.7070)	(0.9405)	(1.0000)	(0.9180)	(0.6732)	(0.4775)
29	0.083899	0.127656	0.212892	0.301352	0.419304	0.614726	0.793001
	(0.4695)	(0.6630)	(0.9095)	(1.0000)	(0.9492)	(0.7172)	(0.5118)
30	0.080370	0.123821	0.209643	0.292947	0.419117	0.618786	0.80458
	(0.4849)	(0.6850)	(0.9285)	(1.0000)	(0.9295)	(0.6867)	(0.4872)

$L(z_i, 29) = (z_i - 29)^2 I(z_i \geq 29) + c_2(29 - z_i)^2 I(z_i < 29)$

Number in parenthesis is the ratio of Monte Carlo p. e. l. associated with settings under the specified asymmetric quadratic loss to Monte Carlo predictive expected loss associated with settings under quadratic loss.

Table 6

Monte Carlo Expected Asymmetric Linear Losses of Sequential Settings

period	$c_4 = 0.1$	$c_4 = 0.2$	$c_4 = 0.5$	$c_4 = 1$	$c_4 = 2$	$c_4 = 5$	$c_4 = 10$
21	0.119324	0.202855	0.382966	0.585670	0.845460	1.263509	1.618714
	(0.4149)	(0.6325)	(0.9117)	(1.0000)	(0.9221)	(0.6614)	(0.4539)
22	0.104290	0.171361	0.307908	0.449873	0.616929	0.852998	1.033929
	(0.4237)	(0.6376)	(0.9145)	(1.0000)	(0.9123)	(0.6293)	(0.4157)
23	0.106925	0.174766	0.312727	0.454977	0.620993	0.853337	1.029807
	(0.4153)	(0.6255)	(0.9058)	(1.0000)	(0.9208)	(0.6403)	(0.4238)
24	0.115537	0.189059	0.338848	0.492498	0.672058	0.921254	1.113741
	(0.4092)	(0.6185)	(0.9018)	(1.0000)	(0.9257)	(0.6458)	(0.4293)
25	0.107619	0.177095	0.318567	0.463654	0.635185	0.881096	1.073249
	(0.4293)	(0.6456)	(0.9225)	(1.0000)	(0.9070)	(0.6248)	(0.4138)
26	0.104980	0.172942	0.312292	0.456854	0.628456	0.862257	1.041813
	(0.4170)	(0.6299)	(0.9107)	(1.0000)	(0.9178)	(0.6301)	(0.4154)
27	0.101903	0.167766	0.312292	0.456854	0.628456	0.862257	1.041813
	(0.4220)	(0.6357)	(0.9129)	(1.0000)	(0.9146)	(0.6249)	(0.4103)
28	0.100859	0.166016	0.297784	0.435606	0.598312	0.826324	0.996735
	(0.4284)	(0.6443)	(0.9180)	(1.0000)	(0.9146)	(0.6249)	(0.4154)
29	0.098707	0.162581	0.292314	0.432911	0.588298	0.811293	0.973696
	(0.4163)	(0.6280)	(0.9018)	(1.0000)	(0.9044)	(0.6225)	(0.4072)
30	0.098184	0.162106	0.292161	0.432308	0.585039	0.806777	0.972511
	(0.4190)	(0.6324)	(0.9064)	(1.0000)	(0.8969)	(0.6149)	(0.4032)

$L(z_i, 29) = (z_i - 29)I(z_i \geq 29) + c_4(29 - z_i)I(z_i < 29)$

Number in parenthesis is the ratio of Monte Carlo p. e. l. associated with settings under the specified asymmetric linear loss to Monte Carlo p. e. l. associated with solutions under absolute loss.

Table 7

Data Generated

t	y_t	y_{t-1}	x_t	e_t
1	25.896328	25.000000	2.759798	0.376732
2	25.180449	25.896328	2.160627	0.142134
3	25.432125	25.180449	2.893026	−0.498287
4	24.782524	25.432125	2.499965	−0.563106
5	22.899178	24.782524	2.019843	−0.966527
6	22.730204	22.899178	2.532884	−0.654905
7	22.852404	22.730204	2.790012	−0.911782
8	22.839865	22.852404	2.340755	−0.123567
9	24.610657	22.839865	3.058722	0.221322
10	24.792631	24.610657	2.380650	0.342806
11	25.869690	24.792631	3.080128	−0.124670
12	26.450392	25.869690	2.605446	0.543747
13	26.829699	26.450392	2.324308	1.020769
14	27.521614	26.829699	3.069192	−0.080527
15	28.002813	27.521614	2.938142	0.109238
16	27.037954	28.002813	2.649849	−0.663994
17	25.630766	27.037954	1.912638	0.175128
18	25.498465	25.630766	2.413801	0.166248
19	25.149786	25.498465	2.324782	0.101449
20	26.405947	25.149786	2.885370	0.515378

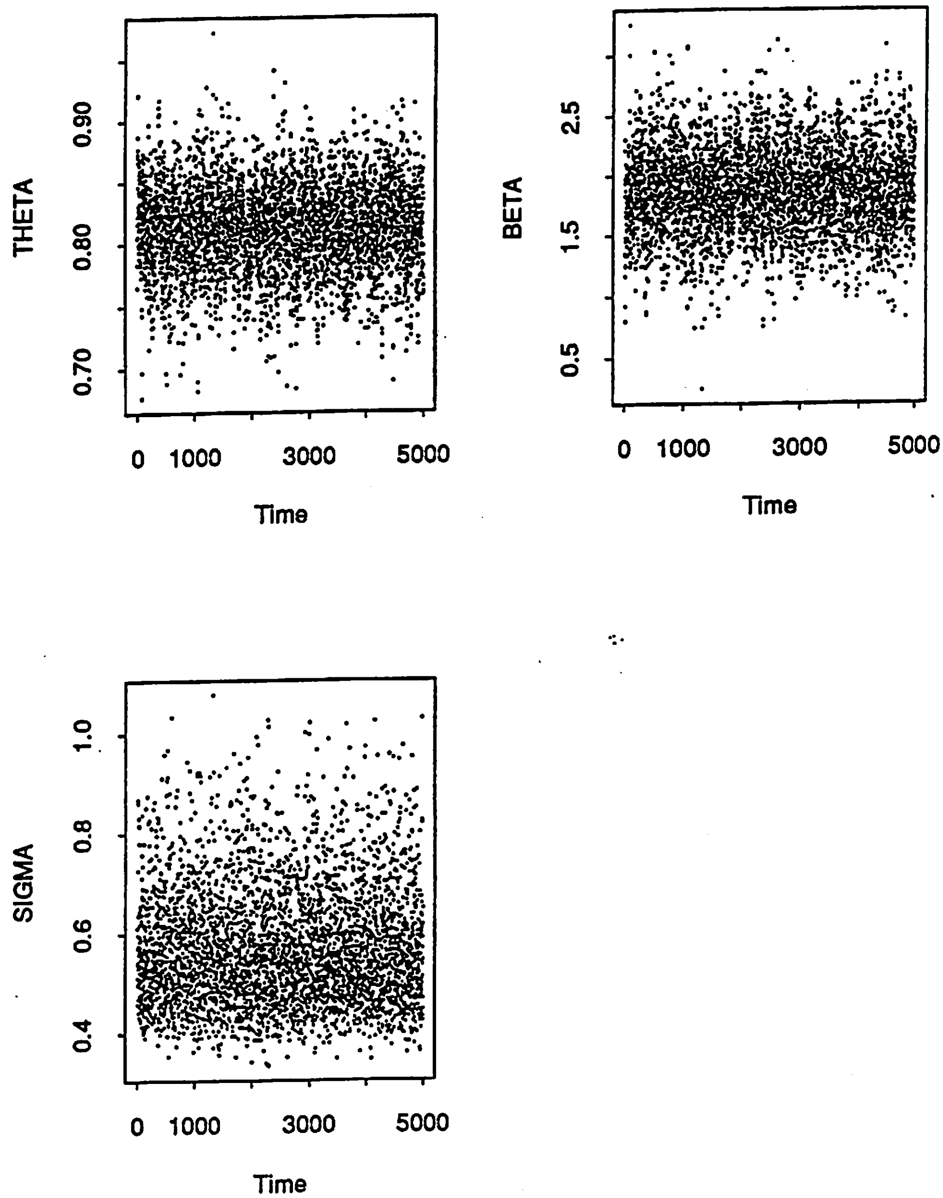

Figure 1: Scatter plots for 5000 subsampled values of θ, β and σ. The Gibbs sampler has a warm-up of 400 iterations and a spacing of 20 .

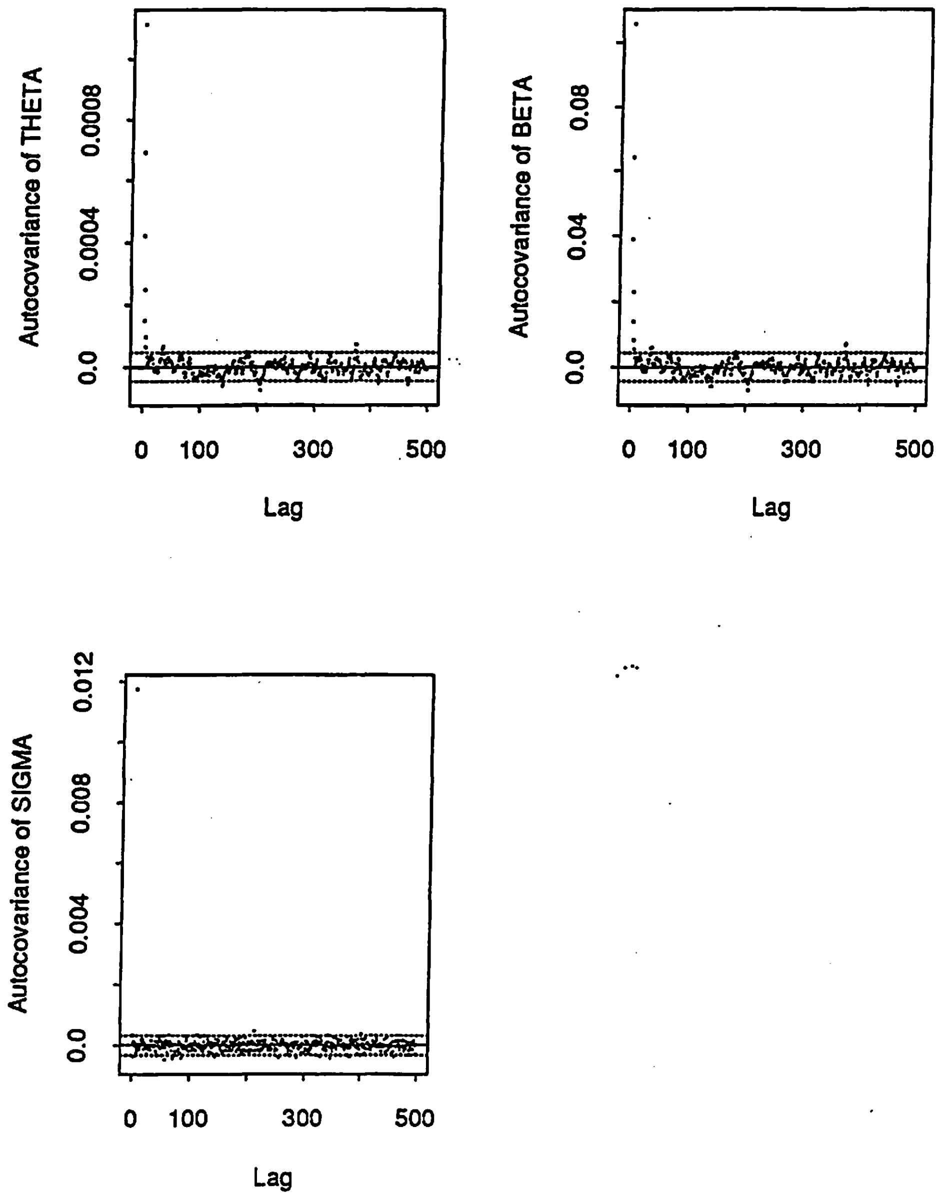

Figure 2: Empirical autocovariance curve for θ, β, and σ. Dashed lines are 95% confidence intervals for autocovariance with large lag obtained from the Bartlett formula (Bartlett, 1946).

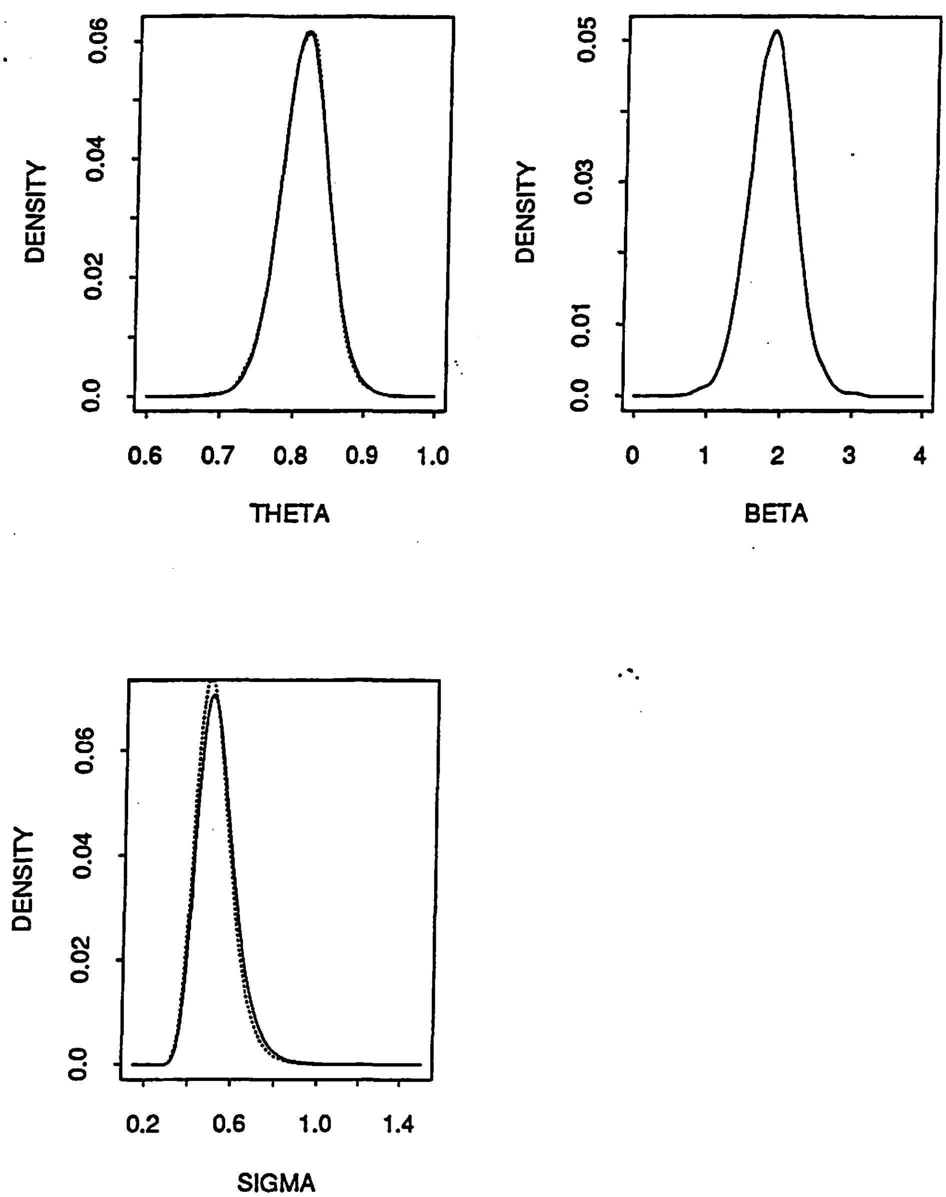

Figure 3: Density estimates for θ, β and σ^{-2}. All dashed lines are density estimates based on full conditional posteriors. The solid line for θ is the exact posterior density, while it is the density estimate based on the reduced conditional posterior for β and σ^{-2}.